State Complexity of Projection on Languages Recognized by Permutation Automata and Commuting Letters

Stefan Hoffmann

University of Trier

International Conference on Developments in Language Theory (DLT) 2021,
August 16 – August 20, 2021
Porto, Portugal

Outline

Notation

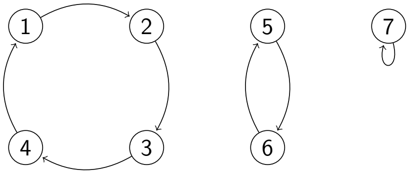
- Deterministc partial automata (PDFA) by $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ as usual.
- Permutation automaton, if $q \mapsto \delta(q, a)$ is a permutation, i.e., bijective mapping, for every $a \in \Sigma$.
- Transformation monoid $\mathcal{T}_{\mathcal{A}}$: Monoid generated by the mappings $q \mapsto \delta(q, a)$, $q \in Q$, for $a \in \Sigma$.
- State complexity of a regularity-preserving operation: largest number of states of an automaton for the result of this operation as a function of the size of automata for the input languages.

Permutation Groups

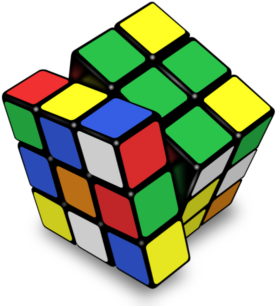
- Permutation groups are subgroups of the set of all permutations.
- Permutation groups model symmetries of objects (via automorphism groups).
- Example: Rubik's cube as a permutation group.

Denote permutations by the cycle notation.

$$(1, 2, 3, 4)(5, 6)(7) \quad =$$



```
      +--------------+
      |              |
      |  1    2    3 |
      |              |
      |  4   top   5 |
      |              |
      |  6    7    8 |
      |              |
+--------------+--------------+--------------+--------------+
|              |              |              |              |
|  9   10   11 | 17   18   19 | 25   26   27 | 33   34   35 |
|              |              |              |              |
| 12  left  13 | 20 front  21 | 28 right  29 | 36  rear  37 |
|              |              |              |              |
| 14   15   16 | 22   23   24 | 30   31   32 | 38   39   40 |
|              |              |              |              |
+--------------+--------------+--------------+--------------+
      |              |
      | 41   42   43 |
      |              |
      | 44 bottom 45 |
      |              |
      | 46   47   48 |
      |              |
      +--------------+
```



Movements: $(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)$
$(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)$
$(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)$
$(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)$
$(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)$
$(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)$

Aim: To a given permutation, find inverse permutation.
Possible strategy: apply sequence of commutators.

Permutation Automata & State Complexity

1. McNaughton (Inf. & Contr. 1967) devised an algorithm for languages recognizable by permutation automata to compute their star-height.
2. Thierrin (Math. Sys. Theo., 1968) investigated right-congruences induced by permutation automata and some closure properties.
3. Hospodár & Mlynárčik (DLT 2020) investigated the state complexity of various operations on permutation automata.
4. Commutative closure was left open in Hospodár & Mlynárčik (DLT 2020). A bound was obtained in Hoffmann (DCFS 2020), but tightness unknown.

Permutation Automata & State Complexity

Operation	Closed?	State Complexity
L^C	Yes	n
$\cap, \cup, \setminus, \oplus$	Yes	nm
KL	No	$m2^n - 2^{n-1} - m + 1$
L^2	No	$n2^{n-1} - 2^{n-2}$
L^*	No	$2^{n-1} + 2^{n-2}$
L^R	Yes	$\binom{n}{\lfloor n/2 \rfloor}$
$L^{-1}K$	Yes	$\binom{m}{\lfloor m/2 \rfloor}, m \leq n$
KL^{-1}	Yes	$m, m \leq n$
$K!L$	No	$(m-1)n + m$
$\text{perm}(L)$	No	$O((n \exp(\sqrt{n \ln n}))^{\lvert\Sigma\rvert})$

The Projection Operation

Definition

Let $\Gamma \subseteq \Sigma$. Then, we define the projection homomorphism $\pi_\Gamma : \Sigma^* \to \Gamma^*$ onto Γ^* by

$$\pi_\Gamma(x) = \begin{cases} x & \text{if } x \in \Gamma; \\ \varepsilon & \text{otherwise;} \end{cases}$$

on the letters $x \in \Sigma$ and set $\pi_\Gamma(\varepsilon) = \varepsilon$ and $\pi_\Gamma(wa) = \pi_\Gamma(w)\pi_\Gamma(x)$ for $w \in \Sigma^*$ and $x \in \Sigma$.

Projection corresponds to a simplified or restricted view of a modelled system (for example observable properties of a discrete event system).

The Projection Operation

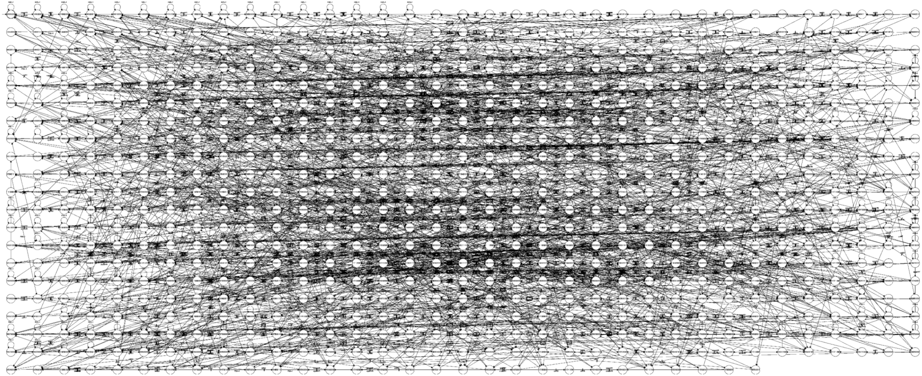


Fig. 1. An example of a simple system G: 729 states, 4400 transitions, 19 events.

Image from Jiráskova & Masopust, *On a Structural Property in the State Complexity of Projected Regular Languages* (2012)

The Projection Operation

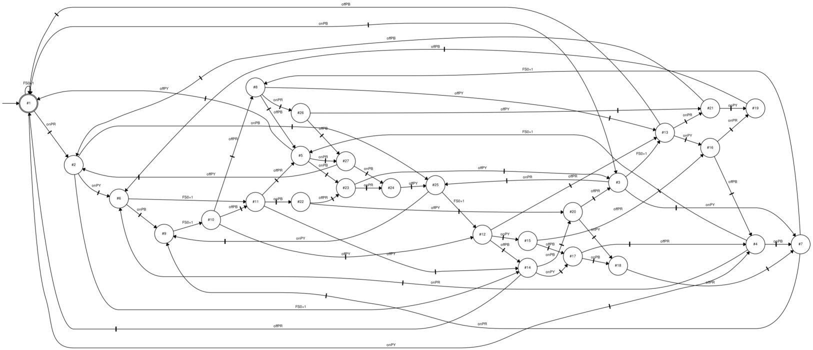


Fig. 2. Projection of *G*: 27 states, 62 transitions, 7 events.

Image from Jiráskova & Masopust, *On a Structural Property in the State Complexity of Projected Regular Languages* (2012)

Projection & State Complexity

The size of a recognizing automaton for a projected language is of interest, as it corresponds to the complexity of algorithms using a simplified view of a modelled system.

- Wong (1998), in the context of discrete event systems, has shown that the projection of a language recognized by an n-state PDFA is recognizable by a PDFA with at most $2^{n-1} + 2^{n-2} - 1$ states and this bound is tight.
- Refined by Jiráskova & Masopust (2012) to the tight bound

$$2^{n-1} + 2^{n-m} - 1$$

  with $m = |\{p, q : p \neq q \text{ and } q \in \delta(p, \Sigma \setminus \Gamma)\}|$ (number of unobservable nonloop transitions) for π_Γ.

Orbits & The Projection Automaton

Definition

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Suppose $\Sigma' \subseteq \Sigma$ and $S \subseteq Q$. The Σ'-orbit of S is the set

$$\mathrm{Orb}_{\Sigma'}(S) = \{\delta(q,u) \mid \delta(q,u) \text{ is defined, } q \in S \text{ and } u \in \Sigma'^*\}.$$

Also, for $q \in Q$, we set $\mathrm{Orb}_{\Sigma'}(q) = \mathrm{Orb}_{\Sigma'}(\{q\})$.

Orbits & The Projection Automaton

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA and $\Gamma \subseteq \Sigma$. Set $\Delta = \Sigma \setminus \Gamma$. Next, we define the projection automaton of $\mathcal{A}$ for Γ as $\mathcal{R}^{\Gamma}_{\mathcal{A}} = (\mathcal{P}(Q), \Gamma, \mu, \mathrm{Orb}_{\Delta}(q_0), E)$ with, for $S \subseteq Q$ and $x \in \Gamma$, the transition function

$$\mu(S, x) = \mathrm{Orb}_{\Delta}(\delta(S, x)) \tag{1}$$

and $E = \{T \subseteq Q \mid T \cap F \neq \emptyset\}$.

Theorem

Let $\mathcal{A}$ be a DFA and $\Gamma \subseteq \Sigma$. Then, $\pi_{\Gamma}(L(\mathcal{A})) = L(\mathcal{R}^{\Gamma}_{\mathcal{A}})$.

Orbits & The Projection Automaton

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA and $\Gamma \subseteq \Sigma$. Set $\Delta = \Sigma \setminus \Gamma$. Next, we define the projection automaton of $\mathcal{A}$ for Γ as $\mathcal{R}_{\mathcal{A}}^{\Gamma} = (\mathcal{P}(Q), \Gamma, \mu, \mathrm{Orb}_{\Delta}(q_0), E)$ with, for $S \subseteq Q$ and $x \in \Gamma$, the transition function

$$\mu(S, x) = \mathrm{Orb}_{\Delta}(\delta(S, x)) \tag{1}$$

and $E = \{T \subseteq Q \mid T \cap F \neq \emptyset\}$.

Theorem

Let $\mathcal{A}$ be a DFA and $\Gamma \subseteq \Sigma$. Then, $\pi_{\Gamma}(L(\mathcal{A})) = L(\mathcal{R}_{\mathcal{A}}^{\Gamma})$.

Definition

An automaton $\mathcal{A}$ is a state-partition automaton, if the set of reachable states of $\mathcal{R}_{\mathcal{A}}^{\Gamma}$ partitions Q.

For state-partition automata, $\pi_{\Gamma}(L(\mathcal{A}))$ is recognizable by an automaton with at most n states.

State Complexity of Projection on Permutation Automata

Theorem

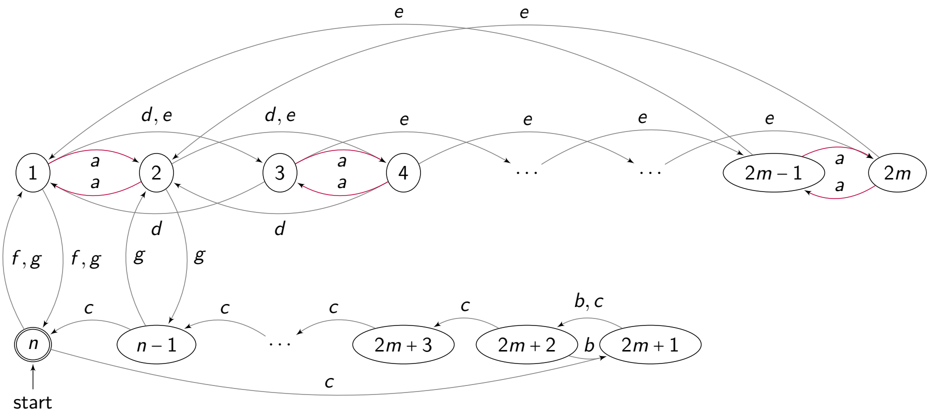
1. $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ *is a permutation automaton.*
2. $\Gamma \subseteq \Sigma$.
3. $m = |\{p, q \in Q : p \neq q, q \in \delta(p, \Sigma \setminus \Gamma)\}|$.

Then:

1. *If* $m = 0$*, then* $\pi_\Gamma(L(\mathcal{A}))$ *is recognizable by an automaton with at most* $|Q|$ *states.*
2. *If* $m > 0$*, then* $\pi_\Gamma(L(\mathcal{A}))$ *is recognizable by an automaton with at most* $2^{|Q| - \lceil \frac{m}{2} \rceil} - 1$ *states.*
3. *These bounds are tight.*

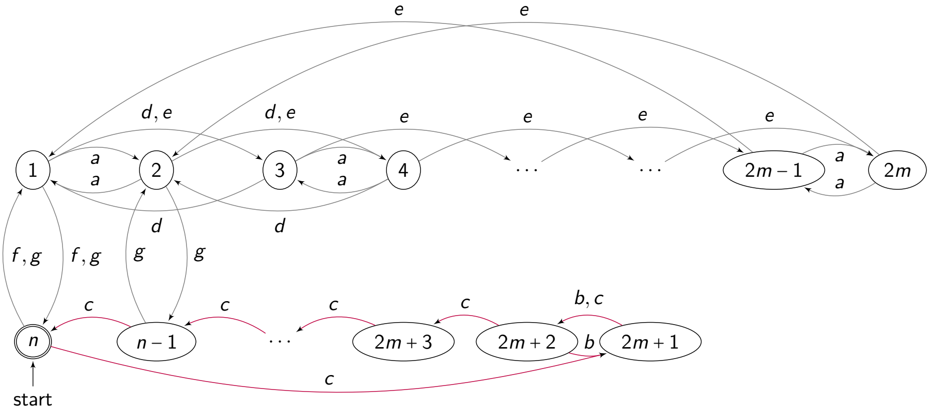
Proof Sketch.

The Γ-orbits partition the state set. Hence, the reachable states of $\mathcal{R}^\Gamma_\mathcal{A}$ are unions of Δ-orbits $\mathrm{Orb}_\Gamma(q)$, $q \in Q$. We show tightness next. □



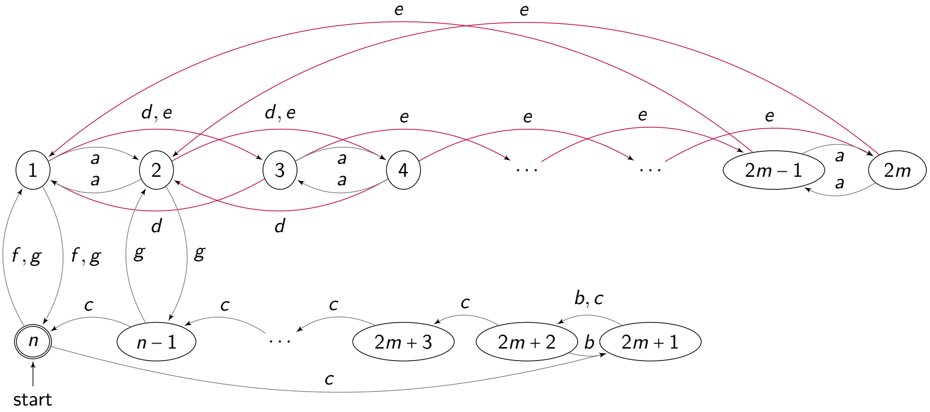
$$
\begin{aligned}
a &= (1,2)(3,4)\cdots(2m-1,2m), \\
b &= (2m+1,2m+2), & c &= (2m+1,2m+2,\ldots,n), \\
d &= (1,3)(2,4), & e &= (1,3,\ldots,2m-1)(2,4,\ldots,2m), \\
f &= (1,n), & g &= (1,n)(2,n-1).
\end{aligned}
$$

$\pi_\Gamma : \Sigma^* \to \Gamma^*$ with $\Gamma = \{b, c, d, e, f, g\}$. Self-Loops omitted.



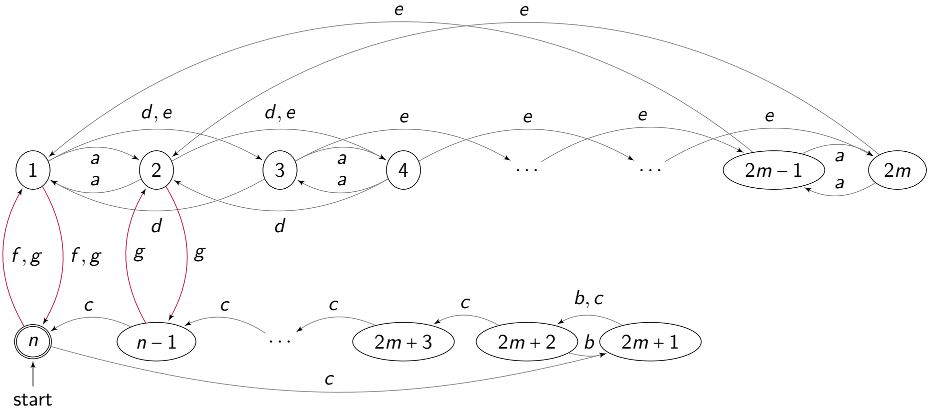
$$
\begin{aligned}
a &= (1,2)(3,4)\cdots(2m-1,2m), \\
b &= (2m+1,2m+2), & c &= (2m+1,2m+2,\ldots,n), \\
d &= (1,3)(2,4), & e &= (1,3,\ldots,2m-1)(2,4,\ldots,2m), \\
f &= (1,n), & g &= (1,n)(2,n-1).
\end{aligned}
$$

$\pi_\Gamma : \Sigma^* \to \Gamma^*$ with $\Gamma = \{b,c,d,e,f,g\}$. Self-Loops omitted.



$$
\begin{aligned}
a &= (1,2)(3,4)\cdots(2m-1,2m), \\
b &= (2m+1, 2m+2), & c &= (2m+1, 2m+2, \ldots, n), \\
d &= (1,3)(2,4), & e &= (1,3,\ldots,2m-1)(2,4,\ldots,2m), \\
f &= (1,n), & g &= (1,n)(2,n-1).
\end{aligned}
$$

$\pi_\Gamma : \Sigma^* \to \Gamma^*$ with $\Gamma = \{b, c, d, e, f, g\}$. Self-Loops omitted.



$$
\begin{aligned}
a &= (1,2)(3,4)\cdots(2m-1,2m), & & \\
b &= (2m+1,2m+2), & c &= (2m+1,2m+2,\ldots,n), \\
d &= (1,3)(2,4), & e &= (1,3,\ldots,2m-1)(2,4,\ldots,2m), \\
f &= (1,n), & g &= (1,n)(2,n-1).
\end{aligned}
$$

$\pi_\Gamma : \Sigma^* \to \Gamma^*$ with $\Gamma = \{b, c, d, e, f, g\}$. Self-Loops omitted.

Normal Subgroups

In a group G, a subgroup N is called normal, if for every $g \in G$ we have $gN = Ng$. In terms of automata:

Definition

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a permutation automaton. Then, a subgroup N of $\mathcal{T}_{\mathcal{A}}$ is called normal, if, for each $\delta_u, \delta_v \in \mathcal{T}_{\mathcal{A}}$ $(u, v \in \Sigma^*)$,

$$(\exists \delta_w \in N : \delta_u = \delta_{wv}) \Leftrightarrow (\exists \delta_{w'} \in N : \delta_u = \delta_{vw'}).$$

Normal Subgroups

Theorem

1. *$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ permutation automaton, $\Gamma \subseteq \Sigma$.*
2. *$N = \{\delta_u : Q \to Q : u \in (\Sigma \setminus \Gamma)^*\}$ normal in $\mathcal{T}_{\mathcal{A}}$.*

Then, $\mathcal{A}$ is a state-partition automaton for π_Γ.

Proof.
The action of the letters is compatible with the orbits for $\Delta = \Sigma \setminus \Gamma$, more precisely $\delta(\mathrm{Orb}_\Delta(q), x) = \mathrm{Orb}_\Delta(\delta(q, x))$. □

Commuting Letters

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be an automaton with n states.

1. When is $\pi_\Gamma(L(\mathcal{A}))$ recognizable by an automaton with at most n states?
2. Hitherto, only state-partition automata and automata recognizing finite languages projected onto unary languages have this property.

The following property of $\Gamma \subseteq \Sigma$ ensures this:

$$\delta(q, ab) = \delta(q, ba)$$

for all $q \in Q$, $a \in \Sigma \setminus \Gamma$, $b \in \Gamma$.

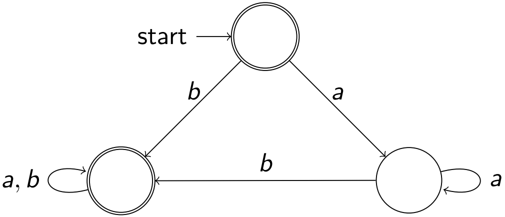
Commuting Letters

Theorem

Suppose $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ is an arbitrary DFA. Let $\Gamma \subseteq \Sigma$ be such that, for each $a \in \Sigma \setminus \Gamma$, $b \in \Gamma$ and $q \in Q$, we have $\delta(q, ab) = \delta(q, ba)$. Then, $\pi_\Gamma(L)$ is recognizable by a DFA with at most $|Q|$ states.

Example

We have genuinely new automata whose projected image has state complexity at most $|Q|$. The following commutative automaton is neither a state-partition automaton, nor does it recognizes a finite language.



Varieties

A variety $\mathcal{V}$ associates with every alphabet Σ a class of regular languages $\mathcal{V}(\Sigma^*)$ over Σ which is a

1. Boolean algebra,
2. closed under left- and right quotients, i.e.,

   $$u^{-1}L = \{v \in \Sigma^* : uv \in L\}, \quad Lu^{-1}L = \{v \in \Sigma^* : vu \in L\}$$

   for $u \in \Sigma^*$, $L \in \mathcal{V}(\Sigma^*)$,
3. closed under inverse homomorphic images.

The class of languages recognized by permutation automata could be seen as a variety.

Varieties

A variety $\mathcal{V}$ associates with every alphabet Σ a class of regular languages $\mathcal{V}(\Sigma^*)$ over Σ which is a

1. Boolean algebra,
2. closed under left- and right quotients, i.e.,

$$u^{-1}L = \{v \in \Sigma^* : uv \in L\}, \quad Lu^{-1}L = \{v \in \Sigma^* : vu \in L\}$$

   for $u \in \Sigma^*$, $L \in \mathcal{V}(\Sigma^*)$,
3. closed under inverse homomorphic images.

The class of languages recognized by permutation automata could be seen as a variety.

Our method of proofs implies the following:

Theorem
Let $\mathcal{V}$ be a variety of commutative languages. If $L \in \mathcal{V}(\Sigma^*)$, then $\pi_\Gamma(L) \in \mathcal{V}(\Gamma^*)$.

Thank you for your attention!

All references could be found in the paper.