Constrained Synchronization and Subset Synchronization Problems for Weakly Acyclic Automata

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International Conference on Developments in Language Theory (DLT) 2021,
August 16 – August 20, 2021
Porto, Portugal

Notation

Let Σ be some finite alphabet.

- Languages: subsets of Σ^* (the free monoid)
- Automata $A = (\Sigma, Q, \delta, s_0, F)$ with input alphabet Σ, state set Q, transition function $\delta : Q \times \Sigma \to Q$, start state s_0 and final states F.
- Semi-Automata: Like automata, but without a start state or final states.
- Regular language: Languages described by finite automata as labels of paths from the start to some final state.

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- $w \in \Sigma^*$ synchronizing if $|\delta(Q, w)| = 1$.
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- SYNC: Given DCSA A and $k \geq 0$, check if A has a synchronizing word of length at most k is NP-complete. (Rystsov 1980, Eppstein 1990)

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- A synchronizing if A admits a synchronizing word. For DCSA, this property can be checked in P, BUT:
- Sync: Given DCSA A and $k \geq 0$, check if A has a synchronizing word of length at most k is NP-complete. (Rystsov 1980, Eppstein 1990)
- Famous combinatorial **conjecture** attributed to Černý: Each synchronizing DCSA has a synchronizing word of length at most $(|Q| - 1)^2$. **Open question** for > 50 years.
- Recent improvements on the leading coefficients of a cubic upper bound: Szykuła STACS 2018, Shitov JALC 2019.

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Fix a partial deterministic finite automaton (PDFA) $\mathcal{B} = (\Sigma, P, \mu, p_0, F)$.

Definition

$L(\mathcal{B})$-Constr-Sync
Input: DCSA $A = (\Sigma, Q, \delta)$.
Question: Is there a synchronizing word w for A with $w \in L(\mathcal{B})$?

History of L-Constr-Sync

- $L(B)$-Constr-Sync introduced in (Fernau et al., MFCS 2019)
- There, a completey classification for small constraint automata with $|Q| \leq 2$ and $|\Sigma| \leq 3$ was given. The problem is, depending on the constraint language, either PSPACE-complete or in P.
- For polycyclic constraint automata, the problem is always in NP (Hoffmann, ICTCS 2020).
- For commutative regular constraint languages, a trichotomy result was achieved (Hoffmann, COCOON 2020); showing that only NP-complete, PSPACE-complete or problems in P arise.
- Complete classifications for $|Q| \leq 3$ obtained (COCOON, 2021). Only problems which are NP-complete, PSPACE-complete, or in P occur.
- For letter-bounded constraint languages, a dichotomy between P and NP-completeness was shown (FCT, 2021).

Weakly Acyclic Automata

Weakly Acyclic Automata (Ryzhikov 2019)

DCSA $A = (\Sigma, Q, \delta)$ is called weakly acyclic, if there exists an ordering $q_1, q_2, \ldots, q_n$ of its states such that if $\delta(q_i, x) = q_j$ for $x \in \Sigma$, then $i \leq j$.

- A DCSA A is weakly acyclic if and only if the only loops are self-loops.
- These automata are also known as acyclic (Jiŕasková & Masopust, 2012) or partially ordered (Brzozowski & Fich, 1980).

This work

Here, we look at $L(\mathcal{B})$-CONSTR-SYNC for weakly acyclic input automata. Formally:

$L(\mathcal{B})$-WAA-CONSTR-SYNC
Input: *Weakly Acyclic Semi-Automaton* $\mathcal{A} = (\Sigma, Q, \delta)$.
Question: *Is there a synchronizing word* w *for* $\mathcal{A}$ *with* $w \in L(\mathcal{B})$?

Overview of Complexity for Different Types of Input Automata

Input Aut. Type	Complexity Class	Hardness	Reference
General Automata	PSPACE	PSPACE-hard	Fernau et al. (2019)
With Sink State	PSPACE	PSPACE-hard	Fernau et al. (2019)
Weakly Acyclic	NP	NP-hard	present work
TTSPL	NP	NP-hard	present work
Simple Idempotents[1]	P for $\|\Sigma\| = 2$ and Constr. Aut ≤ 3 states	-	unpublished
Commutative[2]	P	-	unpublished

The stated hardness results are obtained with the constraint $a(b+c)^*$ in the first four cases.

[1] $\mathcal{A} = (\Sigma, Q, \delta)$ has the property that for every $a \in \Sigma$ either $\delta(Q, a) = Q$ or $|\delta(Q, a)| = |Q| - 1$ and $\delta(q, aa) = \delta(q, a)$ for all $q \in Q$.

[2] $\mathcal{A} = (\Sigma, Q, \delta)$ has the property that for every $a, b \in \Sigma$ and $q \in Q$ we have $\delta(q, ab) = \delta(q, ba)$.

Constrained Synchronization for Weakly Acyclic Automata

Proposition

Let A be a weakly acyclic automaton with n states and $\mathcal{B} = (\Sigma, P, \mu, p_0, F)$ be a fixed PDFA. Then, a shortest synchronizing word $w \in L(\mathcal{B})$ for A has length at most $|P|\binom{n}{2}$.

Theorem

For any PDFA $\mathcal{B}$, we have $L(\mathcal{B})$-WAA-CONSTR-SYNC $\in NP$.

Questions

What precise complexities inside of NP are realizable? Are there constraint automata giving NP-complete problems?

Classification for Small Constraint Automata

Proposition

For the following constraint languages, the constrained synchronization problem for weakly acyclic automata is NP-hard:

$$
\begin{array}{lll}
a(b+c)^* & (a+b+c)(a+b)^* & (a+b)(a+c)^* \\
(a+b)^*c(a+b)^* & a^*b(a+c)^* & a^*(b+c)(a+b)^* \\
a^*b(b+c)^* & (a+b)^*c(b+c)^* & a^*(b+c)(b+c)^*.
\end{array}
$$

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The general problem $L(\mathcal{B})$-Constr-Sync, for the above constraint languages, is PSPACE-complete. However, for the constraint languages

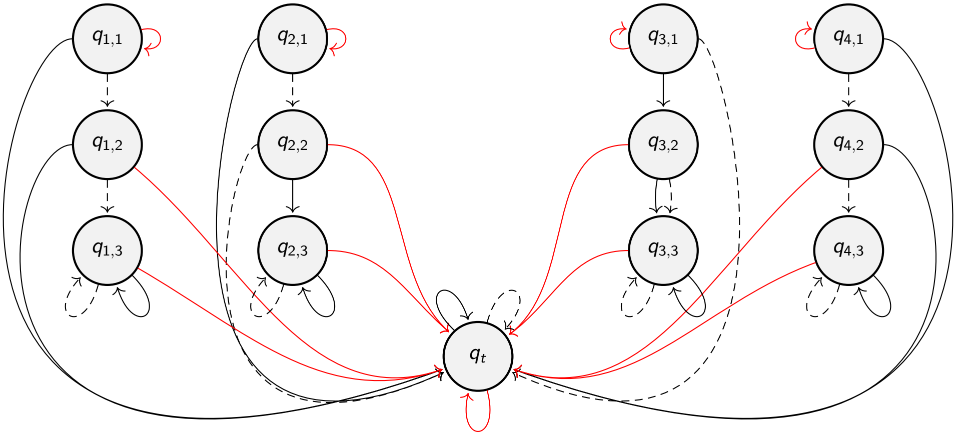
$$((a+b)^*c) \qquad ((a+b)^*ca^*)$$

the general problem is PSPACE-complete, but $L(\mathcal{B})$-WAA-Constr-Sync $\in$ P.

The Reduction (A Modification of the Rystsov-Eppstein Construction)

Example for $c(a+b)^*$ from a SAT instance with two variables, letter c red.

$(x_1 \vee x_2) \quad \wedge \quad (x_1 \vee \neg x_2) \quad \wedge \quad (\neg x_1) \quad \wedge \quad (x_1 \vee x_2)$



Classification for Small Constraint Automata

Classification

For constraint PDFAs $\mathcal{B}$ with at most two states over an at most ternary alphabet, $L(B)$-WAA-Constr-Sync is NP-complete precisely in the cases:

$$\begin{array}{lll} a(b+c)^* & (a+b+c)(a+b)^* & (a+b)(a+c)^* \\ (a+b)^*c(a+b)^* & a^*b(a+c)^* & a^*(b+c)(a+b)^* \\ a^*b(b+c)^* & (a+b)^*c(b+c)^* & a^*(b+c)(b+c)^* \end{array}$$

and polynomial time solvable otherwise.

Subset Synchronization Problems

Definition

SYNC-FROM-SUBSET
Input: $A = (\Sigma, Q, \delta)$ and $S \subseteq Q$.
Question: Is there a word w with $|\delta(S, w)| = 1$?

Definition

SYNC-INTO-SUBSET
Input: $A = (\Sigma, Q, \delta)$ and $S \subseteq Q$.
Question: Is there a word w with $\delta(Q, w) \subseteq S$?

Definition

SETTRANSPORTER
Input: $\mathcal{A} = (\Sigma, Q, \delta)$ and two subsets $S, T \subseteq Q$.
Question: Is there a word $w \in \Sigma^*$ such that $\delta(S, w) \subseteq T$?

Subset Synchronization Problems

- Sync-From-Subset is NP-complete for at least binary fixed Σ, and in P for unary alphabets (Ryzhikov 2019).
- Sync-Into-Subset is in P for any fixed alphabet Σ.
- SetTransporter is NP-complete for at least binary fixed Σ, and in P for unary alphabets.

Two-Terminal Series-Parallel Automata Graphs with Loops

Definition (Fernau & Bruchertseifer, 2019)

A directed (multi-)graph G is two-terminal series-parallel with loops, or TTSPL for short, with terminals s (the source) and t (the sink), if it can be produced by a sequence of the following operations:

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4. Add a loop to a terminal node of a given TTSPL graph.

TTSPL Automata

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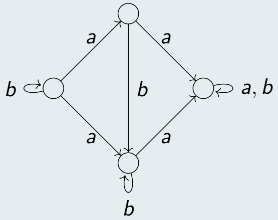
Automata whose automaton graph is a TTSPL graph, i.e., a series-parallel graph with self-loops and two terminals.

Observation

TTSPL automata are a proper subclass of the weakly acyclic automata. For example, the following weakly acyclic automaton is not a TTSPL automaton.

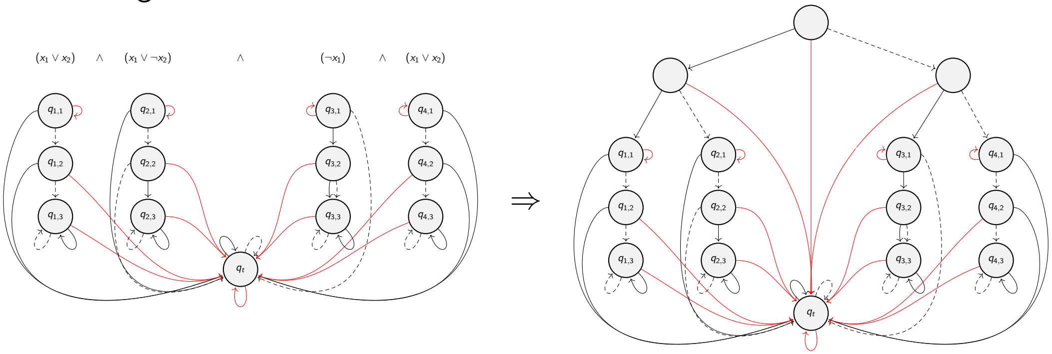


Constrained Synchronization for TTSPL input automata

Natural Question

How do our results relate to this subclass?

Transforming WAAs to TTSPLs:



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So, we find the same complexity classification as before when the problem is restricted to TTSPL automata.